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Dynamics of condensation in zero-range processes

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Abstract
The dynamics of a class of zero-range processes exhibiting a condensation transition in the stationary state is studied. The system evolves in time starting from a random disordered initial condition. The analytical study of the large-time behaviour of the system in its mean-field geometry provides a guide for the numerical study of the one-dimensional version of the model. Most qualitative features of the mean-field case are still present in the one-dimensional system, both in the condensed phase and at criticality. In particular the scaling analysis, valid for the mean-field system at large time and for large values of the site occupancy, still holds in one dimension. The dynamical exponent z, characteristic of the growth of the condensate, is changed from its mean-field value 2 to 3. In the presence of a bias, the mean-field value $z = 2$ is recovered. The dynamical exponent z_c, characteristic of the growth of critical fluctuations, is changed from its mean-field value 2 to a larger value, $z_c \simeq 5$. In the presence of a bias, $z_c \simeq 3$.

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1. Introduction

In a recent work, Kafri *et al* [1] gave a criterion for the existence of phase separation in driven one-dimensional systems. Discussions on this theme have been the subject of a long series of investigations [2–17]. The criterion relies on a mapping, in the stationary state, of the system onto a zero-range process.

For example, consider a two-species model, where each site of a one-dimensional ring is either occupied by a (+) particle, or by a (−) particle, or is vacant (0). Positive particles are driven to the right, negative particles to the left, and they exchange positions when meeting. Define domains as uninterrupted sequences of (+) or (−) particles. Then identify the zeros with boxes, the particles with balls, and a domain with the content of the box attached to the zero to its left, say. Then the system of (±) particles can be viewed as a system of balls in boxes, where the rate at which a ball leaves a box is given by the current out of a domain. This process is known as a zero-range process (ZRP) [18–20] , more precisely defined below.

Note that such a mapping of the particle system to the corresponding ZRP gives a coarse-grained description of the former, because the spatial structure inside domains is lost. Once the hopping rate d_k out of a box containing k balls is known, the analysis of the stationary state of the ZRP is straightforward. Depending on the behaviour of d_k at large k, it is easy to show that there is, or there is not, a condensation transition, where a macroscopic fraction of balls is contained in one box, the condensate. This corresponds in the original particle system to a condensed phase where a single domain contains a macroscopic fraction of the particles.

This is, in essence, the analysis given in [1]. Examples of particle systems with two species satisfying the criterion for the existence of phase separation have been subsequently given in [21].

These considerations concern the stationary state. We are thus naturally led to address the question of the dynamics, both for the underlying original microscopic particle models, and for their coarse-grained counterpart, the ZRP.

For the latter, the agenda is clear. The question is to determine the temporal evolution of the system starting from a random initial condition, once given the hopping rate out of a box. The present paper is devoted to the study of the dynamics of a class of ZRP giving rise to a condensation transition in their stationary state. We will proceed in two steps. We study first the system in its mean-field geometry, where all boxes are connected. This gives then a framework for the study of the one-dimensional system.

The dynamics of the microscopic models defined in [21], or more generally of two-species models satisfying the criterion for phase separation, is more subtle because it involves two (*a priori* different) scales of time: the intra-domain dynamics, corresponding to the equilibration inside a domain, and the inter-domain dynamics, corresponding to the coarsening of domains. One can therefore wonder whether during its temporal evolution the particle system is equivalent to a ZRP with constant hopping rate. This study is postponed to another publication [22].

The aim of this paper is therefore twofold. Firstly it addresses the question of the dynamics of zero-range processes *per se*. Indeed, ZRP are sufficiently ubiquitous in the field of stochastic processes to warrant the interest of such a study. For instance, the analysis of the zeta urn model [23–25] is surprisingly close to that presented in the next sections. Secondly it provides a template for the study of the dynamics of particle systems, whenever these can be mapped, at least at a coarse-grained level, to a ZRP, as explained above.

2. ZRP: first definitions

2.1. Mean-field geometry and equilibrium: general formalism

A generic zero-range process is defined as follows. Consider a set of M sites (or boxes), $i = 1, \ldots, M$, located on a lattice of arbitrary dimension. The process may as well be defined on the complete graph, that for short we will refer to as the mean-field geometry, where all sites (boxes) are connected. In the present study we will consider both ZRP in the mean-field geometry (section 3) and on the one-dimensional lattice (section 4). In each box we have N_i indistinguishable particles such that

$$\sum_{i=1}^{M} N_i = N.$$

The dynamics of the system is given by the rates at which a particle leaves the departure box d, containing $N_d = k$ particles, and is put in the arrival box a containing $N_a = l$ particles. The hopping rate, denoted by W_{kl}, can *a priori* depend both on k and l. By definition of a ZRP, one chooses the rate W_{kl} to depend on the occupation of the departure box only. However before doing so, we continue considering a generic rate W_{kl} for a while.

A configuration of the system is specified by the occupation numbers $N_i(t)$, hence a complete knowledge of its dynamics involves the determination of $\mathcal{P}(N_1, N_2, \ldots, N_M)$, the probability of finding the system in a given configuration at time t. Hereafter, we will only focus our attention on the probability of finding k particles in the generic box $i = 1$,

$$f_k(t) = \mathcal{P}(N_1(t) = k)$$

that is, the marginal distribution of the occupation number $N_1(t)$ of this box. Conservation of probability and of density yields

$$\sum_{k=0}^{\infty} f_k(t) = 1 \tag{1}$$

$$\sum_{k=1}^{\infty} k f_k(t) = \rho \tag{2}$$

where we have taken the thermodynamic limit $N \to \infty, M \to \infty$, with fixed density $\rho = N/M$.

Suppose now that the system has the mean-field geometry, i.e., all the boxes are connected. In this case, the temporal evolution of the occupation probability $f_k(t)$ is explicitly given by the master equation

$$\frac{\mathrm{d} f_k(t)}{\mathrm{d}t} = \mu_{k+1} f_{k+1} + \lambda_{k-1} f_{k-1} - (\mu_k + \lambda_k) f_k \qquad (k \geqslant 1)$$

$$\frac{\mathrm{d} f_0(t)}{\mathrm{d}t} = \mu_1 f_1 - \lambda_0 f_0$$

where

$$\mu_k = \sum_{l=0} W_{kl} f_l \tag{3}$$

$$\lambda_k = \sum_{l=1} W_{lk} f_l \tag{4}$$

are respectively the rates at which a particle leaves box number 1 ($N_1 = k \to N_1 = k - 1$), or arrives in this box ($N_1 = k \to N_1 = k + 1$). This is the master equation for a random walk for N_1, i.e., on the positive integers $k = 0, 1, \ldots$. The equation for f_0 is special because one cannot select an empty box as a departure box, nor can N_1 be negative. The conservation law $\mathrm{d}\sum k f_k/\mathrm{d}t = 0$, valid at all times, implies the sum rule

$$\sum_{k=1}^{\infty} \mu_k f_k = \sum_{k=0}^{\infty} \lambda_k f_k. \tag{5}$$

In the stationary state ($\dot{f}_k = 0$), we find

$$\frac{f_{k+1,\mathrm{eq}}}{f_{k,\mathrm{eq}}} = \frac{\lambda_k}{\mu_{k+1}}$$

which expresses the detailed balance condition at equilibrium, yielding

$$f_{k,\mathrm{eq}} = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} f_{0,\mathrm{eq}}$$

where $f_{0,\mathrm{eq}}$ is fixed by normalization.

We now come back to a genuine ZRP, where the rate W_{kl} only depends on the occupation of the departure box. We denote the hopping rate out of the departure box by $W_{kl} = d_k$. (Alternatively, one can consider processes where W_{kl} only depends on the occupation of the arrival box, as for the zeta urn model [23, 24]: $W_{kl} = a_l$ is the hopping rate *into* the arrival box; see below.) We thus have

$$\mu_k = d_k \qquad \lambda_k = \sum_{l=1}^{\infty} d_l f_l \equiv \bar{d}_t. \tag{6}$$

Note that (5) is automatically satisfied. We obtain

$$f_{k,\mathrm{eq}} = \frac{p_k z^k}{\sum_{k=0}^{\infty} p_k z^k} \tag{7}$$

where $z \equiv \sum d_l f_{l,\mathrm{eq}} = \bar{d}_{\mathrm{eq}}$, and

$$p_k = \frac{1}{d_1 \cdots d_k} \qquad (k \geqslant 1) \qquad p_0 = 1. \tag{8}$$

It is important to emphasize here that the results above, equations (7) and (8), also hold for the equivalent one-dimensional ZRP in the stationary state. Indeed a well-known property of ZRP states that in the stationary state the probability $\mathcal{P}(N_1, N_2, \ldots, N_M)$ is given by a product measure [18, 19]. Explicitly $\mathcal{P}(N_1, N_2, \ldots, N_M)$, with $\sum N_i = N$, factorizes into the product $f(N_1) f(N_2) \cdots f(N_M)$, where each factor $f(N_i = k)$ is equal to $f_{k,\mathrm{eq}}$ as given by (7) and (8). For an infinite system this factorization property holds at all time in the mean-field geometry, by essence.

2.2. *Class of ZRP with a condensation transition*

Let us apply this formalism to the case where, at large k

$$d_k \approx 1 + \frac{b}{k} + \cdots. \tag{9}$$

Then it is easy to see that, in the same regime, $p_k \sim k^{-b}$. Hence the series $\sum_{k=0}^{\infty} p_k z^k$ has a radius of convergence $z_c = 1$. The unknown parameter $z \equiv \bar{d}_{\mathrm{eq}}$ is determined by using (2), which yields

$$\sum_{k=1}^{\infty} k f_{k,\mathrm{eq}} = \frac{\sum_{k=0}^{\infty} k p_k z^k}{\sum_{k=0}^{\infty} p_k z^k} = \rho. \tag{10}$$

This expression is monotonically increasing with z. When z reaches the value $z_c = 1$, the right side is converging only if $b > 2$. Therefore we have the following two cases:

- If $b < 2$, equation (10) possesses a solution $z(\rho)$ for any value of ρ. The occupation probabilities are exponentially decaying as

$$f_{k,\mathrm{eq}} = \frac{p_k z^k}{\sum_{k=0}^{\infty} p_k z^k} \sim k^{-b} \exp\left(-\frac{k}{\xi}\right) \qquad \text{(fluid)} \tag{11}$$

  where the correlation length $\xi = |\ln z|^{-1}$. The system is in a fluid phase.
- If $b > 2$, there exists a critical value of the density, where the correlation length diverges, attained when $z = z_c = 1$:

$$\rho_c = \frac{\sum_{k=0}^{\infty} k p_k}{\sum_{k=0}^{\infty} p_k}. \tag{12}$$

This is the equation of a transition line. In other terms, if $\rho < \rho_c$, then (10) has a solution $z(\rho)$, in which case the system is in the fluid phase described above (see equation (11)). Otherwise, if $\rho > \rho_c$, then necessarily $z \equiv \bar{d}_{\text{eq}} = 1$. The system is composed of a critical part and a condensate. The critical part is described by the algebraically decaying distribution

$$f_{k,\text{eq}} = \frac{p_k}{\sum_{k=0}^{\infty} p_k} \sim k^{-b} \qquad \text{(critical)} \tag{13}$$

contributing to ρ_c, as shown by (12). The condensate, contributing to the remaining $\rho - \rho_c$, corresponds to particles sitting in a single box. At criticality ($\rho = \rho_c$) the condensate disappears and (13) still holds.

Let us illustrate the above formalism for the explicit choice of hopping rate

$$d_k = 1 + \frac{b}{k} \tag{14}$$

which is used in the numerical studies of the forthcoming sections. We have

$$f_{k,\text{eq}} = (b-1)\frac{\Gamma(b)\Gamma(k+1)}{\Gamma(k+b+1)} \underset{k\to\infty}{\approx} (b-1)\Gamma(b)k^{-b} \tag{15}$$

and

$$\rho_c = \frac{1}{b-2} \tag{16}$$

which is the equation of the critical line in the b–ρ plane. It is simple to check using (15) that we have consistently $\bar{d}_{\text{eq}} = 1$, both at criticality and in the condensed phase[1].

3. ZRP: dynamics in the mean-field geometry

From now on, we consider a hopping rate d_k out of the departure box of the form (9), i.e., such as $d_k \approx 1 + b/k$ at large k, and for the sake of analytical or numerical illustrations we will choose the explicit form (14), unless specified.

For an infinite system in the mean-field geometry, the knowledge of the occupation probabilities $f_k(t)$ provides a complete description of its dynamics. We consider a system with a Poissonian initial distribution of occupation probabilities,

$$f_k(0) = \mathrm{e}^{-\rho}\,\frac{\rho^k}{k!}$$

i.e., such that initially particles are distributed at random amongst boxes. The temporal evolution of the occupation probabilities f_k is given by the master equation

$$\begin{aligned} \frac{\mathrm{d}f_k(t)}{\mathrm{d}t} &= d_{k+1}f_{k+1} + \bar{d}_t f_{k-1} - (d_k + \bar{d}_t)f_k \qquad (k \geqslant 1) \\ \frac{\mathrm{d}f_0(t)}{\mathrm{d}t} &= d_1 f_1 - \bar{d}_t f_0. \end{aligned} \tag{17}$$

This set of equations is non-linear because $\bar{d}_t \equiv \sum d_l f_l$ is itself a function of the $f_k(t)$. Hence there is no explicit solution of these equations in closed form. Yet one can extract from them an analytical description of the dynamics of the system at long times, both in the condensed phase, and at criticality. The structure of the reasoning borrows from former studies on urn models [23, 24]. (For a review, see [26].)

As we show below, there exist two different regimes in the evolution of the system, both in the condensed phase or at criticality, which we study successively.

[1] Discussions of the statics of condensation in similar models can be found in [20, 25].

3.1. Condensed phase ($\rho > \rho_c$)

Since $\bar{d}_{\mathrm{eq}} = 1$, we set, for large times,

$$\bar{d}_t \approx 1 + A\varepsilon_t \tag{18}$$

where the small time scale ε_t is to be determined. As shown below, $\varepsilon_t \sim t^{-\frac{1}{2}}$.

Regime I (k fixed, t large)**.** For t large enough, boxes empty (d_k) faster than they fill ($\bar{d}_t$). In this regime there is convergence to equilibrium, hence we set

$$f_k(t) \approx f_{k,\mathrm{eq}}(1 + v_k\varepsilon_t) \tag{19}$$

with $f_{k,\mathrm{eq}}$ given by (13), and where the v_k are unknown. This expression carried into (17) yields the stationary equation $\dot{f}_k = 0$, because the derivative $\dot{f}_k$, proportional to $\dot{\varepsilon}_t$, is negligible compared to the right-hand side. We thus obtain the detailed balance condition:

$$\frac{f_{k+1,\mathrm{eq}}}{f_{k,\mathrm{eq}}}\frac{1 + v_{k+1}\varepsilon_t}{1 + v_k\varepsilon_t} = \frac{1 + A\varepsilon_t}{d_{k+1}}.$$

Using (13) and (8), we obtain, at leading order in ε_t, $v_{k+1} - v_k = A$, and finally

$$v_k = v_0 + kA. \tag{20}$$

At this stage, v_0 and the amplitude A are still to be determined.

Regime II (k and t are simultaneously large)**.** This is the scaling regime, with scaling variable $u = k\varepsilon_t$. Following the treatment of [23], we look for a similarity solution of (17) of the form

$$f_k(t) \approx \varepsilon_t^2 g(u). \tag{21}$$

We thus obtain for $g(u)$ the linear differential equation

$$g''(u) + \left(\frac{u}{2} - A + \frac{b}{u}\right) g'(u) + \left(1 - \frac{b}{u^2}\right) g(u) = 0$$

and the result $\varepsilon_t \sim t^{-\frac{1}{2}}$. This is precisely the differential equation found in [23, 24] for the zeta urn model (see the discussion at the end of section 3). The amplitude A can be determined by the fact that the equation has an acceptable solution $g(u)$ vanishing as $u \to 0$ and $u \to \infty$ [23]. It is a universal quantity, only depending on the value of b. The normalization of the solution is fixed by the sum rule (1) yielding [23]

$$v_0 + A\rho_c = -\int_0^\infty \mathrm{d}u\, g(u). \tag{22}$$

The sum rule (2) leads to [23]

$$\int_0^\infty \mathrm{d}u\, u g(u) = \rho - \rho_c. \tag{23}$$

The differential equation above has no closed form solution. However, further information on the form of the solution $g(u)$ can be found in [23, 24]. In particular $g(u)/(\rho - \rho_c)$ is universal in the sense that it only depends on the parameter b (see the discussion below).

An intuitive description of the dynamics of the condensate in the scaling regime is as follows. The typical occupancy k_{cond} of the boxes making the condensate scales as $t^{\frac{1}{2}}$. The total number of particles in the condensate is equal to $M(\rho - \rho_c)$, the remaining $M\,\rho_c$ lying in the fluid. Therefore the number of boxes belonging to the condensate scales as $M(\rho - \rho_c)t^{-\frac{1}{2}}$.

3.2. Criticality ($\rho = \rho_c$)

The analysis follows closely that done in [24]. We just sketch the method here.

We set

$$\bar{d}_t \approx 1 + A\varepsilon_t$$

with $\varepsilon_t = t^{-\omega}$, where the exponent ω is to be determined, and we consider the same two regimes as above. In regime I, we still set (19) for $f_k(t)$. The reasoning leading to the relationship $v_k = v_0 + kA$ (see (20)) is still valid here. In regime II, we look for a similarity solution to (17) of the form

$$f_k(t) \approx f_{k,\mathrm{eq}}\, g_c(u) \qquad u = kt^{-\frac{1}{2}}. \tag{24}$$

The sum rules (1) and (2) lead respectively to the following equations, provided that $b > 3$,

$$v_0 + A\rho_c = 0 \tag{25}$$

$$t^{-\omega}(v_0\rho_c + A\mu_c) = t^{-(b-2)/2}(b-1)\Gamma(b)\int_0^\infty \mathrm{d}u\, u^{1-b}(1 - g_c(u)) \tag{26}$$

where, using (15), $\mu_c = \sum k^2 f_{k,\mathrm{eq}} = (b+1)(b-2)^{-1}(b-3)^{-1}$. Equation (26) fixes the value of ω:

$$\omega = (b-2)/2. \tag{27}$$

The differential equation obeyed by $g_c(u)$ is obtained by carrying (24) into (17). It reads

$$g_c''(u) + \left(\frac{u}{2} - \frac{b}{u}\right) g_c'(u) = 0$$

the solution of which is, with $g_c(0) = 1$,

$$g_c(u) = \frac{2^{-b}}{\Gamma\left(\frac{b+1}{2}\right)} \int_u^\infty \mathrm{d}y\, y^b\, \mathrm{e}^{-y^2/4}. \tag{28}$$

The fall-off of $g_c(u)$ for $u \gg 1$ is very fast: $g_c(u) \sim \exp(-u^2/4)$, hence $f_k(t) \sim \exp(-k^2/4t)$. We finally obtain

$$A = \frac{(b-1)\Gamma(b)}{\mu_c - \rho_c^2} \int_0^\infty \mathrm{d}u\, u^{1-b}(1 - g_c(u)) = \frac{(b-2)(b-3)}{b-1}\Gamma\left(\frac{b}{2}\right).$$

3.3. Universality

To summarize, for any hopping rate of the form $d_k \approx 1 + b/k$, the scaling functions, $g(u)$ in the condensed phase (more precisely, $g(u)/(\rho - \rho_c)$) and $g_c(u)$ at criticality, are universal. In both cases the scaling variable is $u = kt^{-1/2}$. The critical density ρ_c, and, as a consequence, any quantity depending on ρ_c, such as the amplitude v_0, are non-universal, with values depending on the precise definition of d_k. As noted above, the amplitude A is a universal quantity in the condensed phase.

Universality can be checked by comparing the results obtained by a numerical integration of the master equation (17), for different choices of the rate d_k. Besides the form $d_k = 1 + b/k$ of equation (14) we investigated the two following expressions

$$d_k = 1 + \frac{b}{k} + \frac{c}{k^2} \tag{29}$$

$$d_k = \left(1 + \frac{1}{k}\right)^b \tag{30}$$

as well as the form $d_k = 1 + b/(c + k)$.

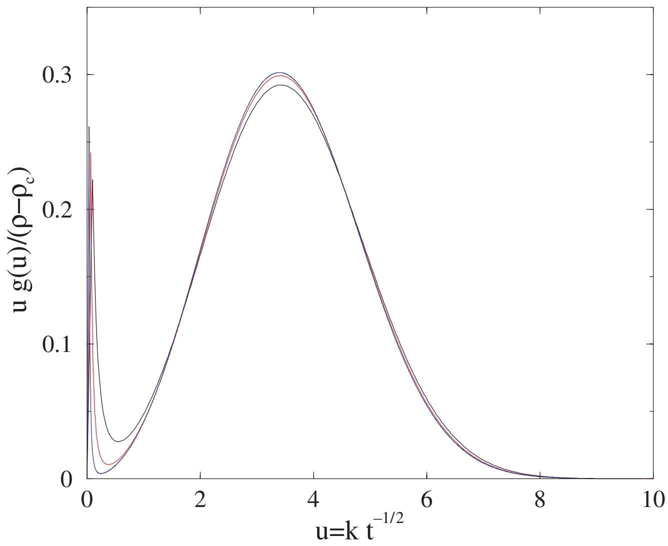

Figure 1. Mean-field geometry: normalized scaling function $ug(u)/(\rho - \rho_c)$ in the condensed phase, for times equal to 10^3, 10^4 and 10^5 (from top to bottom in the dip near the origin); $b = 4$, $\rho = 20\rho_c = 10$.

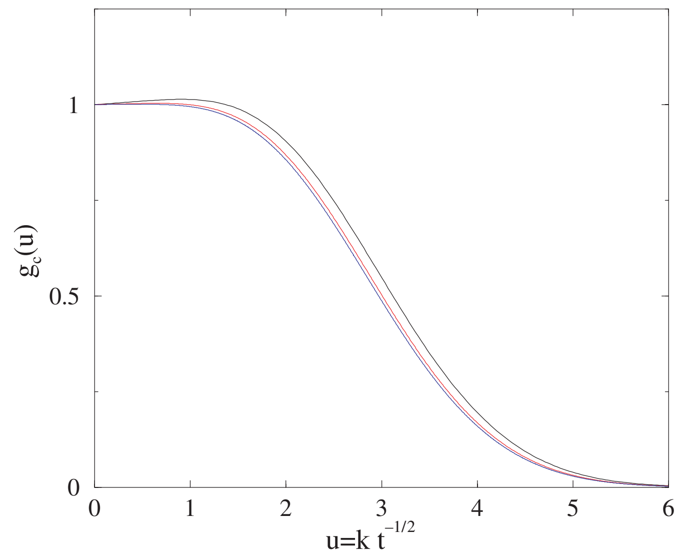

Figure 2. Mean-field geometry: scaling function $g_c(u)$ at criticality for times equal to 10^3, 10^4 and 10^5 (from right to left); $b = 4$, $\rho = \rho_c = 0.5$.

Figure 1 depicts the normalized scaling function $ug(u)/(\rho - \rho_c)$ in the condensed phase, for times equal to 10^3, 10^4 and 10^5. It was obtained with the choice of rate (14), with $b = 4$, and for a density $\rho = 20\rho_c$. The curves obtained with the choice (29) and $c = 10$, or with (30), are hardly distinguishable from the former one. Figure 2 depicts $g_c(u)$ at criticality, for the same range of times, with the choice (14), and $b = 4$. Again, the curves obtained with the choice (29) and $c = 10$, or with (30), are indistinguishable from the former one. The limiting curve of figure 2 coincides exactly with the form (28).

One also finds, for $\rho > \rho_c$ and $b = 4$, $A \simeq 1.9$ and $\int \mathrm{d}u g(u)/(\rho - \rho_c) \simeq 0.336$, which are universal quantities. Correspondingly, for d_k given by (14), and $\rho = 20\rho_c$, $v_0 \simeq -4.15$.

It is remarkable that the universality of the scaling functions $g(u)$ and $g_c(u)$ extends beyond the case of the ZRP under study. Indeed, for the zeta urn model, defined by a rate W_{kl} only depending on the occupation l of the arrival box, much of the formalism above is the same [23, 24]. An intuitive explanation of this fact comes from the comparison of the rates of the two models, at large values of time and occupation numbers. Consider the rates μ_k and λ_k defined in equations (3) and (4). For the ZRP studied here these rates are given by (cf (6))

$$\mu_k = d_k \approx 1 + \frac{b}{k} \qquad \lambda_k = \sum d_l f_l \equiv \bar{d}_t \approx 1 + A\varepsilon_t \qquad \text{(ZRP)}.$$

For the zeta urn model (see [23, 24] for detailed definitions), we have

$$\mu_k = 1 \qquad \lambda_k \approx \left(1 - \frac{\beta}{k}\right)(1 + A\varepsilon_t) \qquad \text{(zeta)}$$

where β is the inverse temperature, playing the role of the parameter b in the model. Simple inspection of the last two equations demonstrates the formal analogy between the two models. This analogy is still stronger if one chooses the rate d_k given by (30). Then, at equilibrium, the formalisms for the corresponding ZRP and for the zeta urn model coincide.

4. ZRP: dynamics in one dimension

The framework of analytic results obtained in the mean-field case provides a guide for the numerical study of the one-dimensional system. It turns out that most of the features of the former survive in one dimension. In particular, the two regimes of time described in the previous section are still present in the one-dimensional case. The main difference lies in the values of the dynamical exponent z characteristic of the growth of the condensate, and of the critical dynamical exponent z_c characteristic of the growth of critical fluctuations. In the condensed phase ($\rho > \rho_c$), numerical simulations, presented hereafter, give evidence for $z = 3$ instead of $z = 2$ in the mean-field geometry. At criticality $z_c \simeq 5$ instead of $z_c = 2$ in the mean-field geometry.

We consider a system on a ring. Sites have multiple occupancy, given by the occupation numbers $N_i(t)$, where $i = 1, \ldots, M$. At each time step a site i is chosen at random and, if not empty, a particle of this site is transferred to one of the two neighbouring sites, to the right or to the left with equal probability, with a rate d_k only depending on the number of particles $N_i = k$, present on site i. As above, we consider a hopping rate such that $d_k \approx 1 + b/k$ at large k. The numerical simulations described below were performed with the choice of rate given by (14).

4.1. Condensed phase ($\rho > \rho_c$)

We begin with the determination of the dynamical exponent z. In regime II (k and t simultaneously large), let us assume the scaling form, inspired by equation (21),

$$f_k(t) \approx \frac{1}{t^{2/z}} g\left(\frac{k}{t^{1/z}}\right).$$

Then, at large time, the mean squared occupation

$$\mu_2 = \langle N_1^2(t) \rangle = \sum_{k=1}^{\infty} k^2 f_k(t) \tag{31}$$

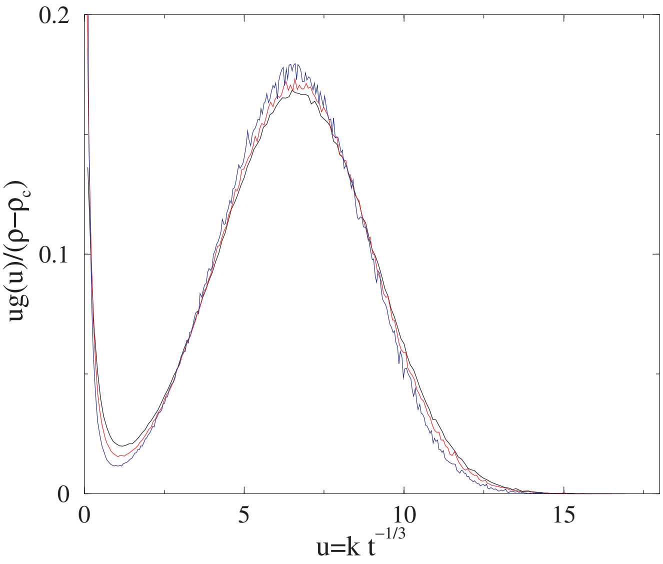

Figure 3. One-dimensional system: normalized scaling function $ug(u)/(\rho - \rho_c)$ in the condensed phase for times equal to 10^3, 3×10^3, and 10^4 (from top to bottom in the dip near the origin); $b = 4$, $\rho = 20\rho_c = 10$.

should behave as

$$\mu_2 \sim t^{1/z} \int_0^\infty \mathrm{d}u\, u^2 g(u). \tag{32}$$

This quantity is easy to measure, and remains accurate even for relatively large times. More generally the moment of order n, $\mu_n = \langle N_1^n(t)\rangle$, scales as $t^{(1/z)(n-1)}$. The behaviour (32) is very well observed in numerical simulations, with $z = 3$ (see figure 4). Figure 3 depicts the normalized scaling function $ug(u)/(\rho - \rho_c)$ for times equal to 10^3, 3×10^3 and 10^4, with $u = kt^{-1/3}$. The data are for $b = 4$ and $\rho = 20\rho_c = 10$.

We checked that the observed value of the dynamical exponent $z = 3$ was robust to changes in b. On the other hand, while for the mean-field case the scaling function $g(u)/(\rho - \rho_c)$ was independent of the density ρ, in the one-dimensional case numerical simulations show dependence on the density. However, this dependence can be absorbed in a rescaling of time,

$$t \longrightarrow (\rho - \rho_c)t \tag{33}$$

as demonstrated by figure 4. It depicts a plot of $\mu_2/(\rho - \rho_c)$ against $((\rho - \rho_c)t)^{1/3}$, for $\rho = 5, 10, 20$. In the linear part of the curves, the slopes are all approximately equal to 3, in very good agreement with the measured numerical values of the integral on the right-hand side of (32), divided by $(\rho - \rho_c)^{4/3}$. Coming back to figure 3, plotting $ug(u)(\rho - \rho_c)^{-2/3}$ against $k((\rho - \rho_c)t)^{-1/3}$ for several densities leads to data collapse, confirming the rescaling of time by the density.

We then determined the characteristics of regime I (k fixed, t large), for $b = 4$ and $\rho = 10$. We first observed that equations (18), (19) and (20) still hold. This was checked in a number of ways. For instance, one can plot $(f_0/f_{0,\mathrm{eq}} - 1)^{-1}$ and $(\bar{d}_t - 1)^{-1}$ against μ_2, yielding straight lines of slopes respectively equal to $v_0 I \simeq -136$ and $AI \simeq 59$. This leads to $A \simeq 0.97$ and $v_0 \simeq -2.2$. These values are consistent with relation (22), since the measured value of the integral $\int \mathrm{d}u\, g(u) \simeq 1.7$. The amplitude A also depends on ρ, with values equal respectively

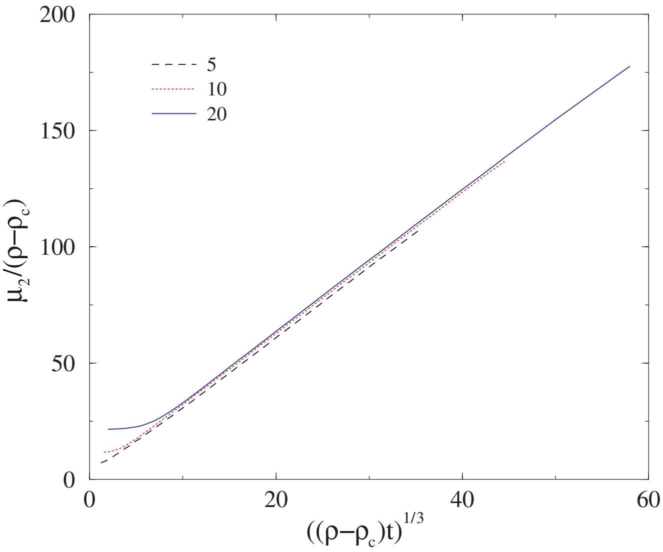

Figure 4. One-dimensional system: rescaled mean squared occupation $\mu_2/(\rho - \rho_c)$ in the condensed phase, for three values of the density: $\rho = 5, 10, 20$, from bottom to top; $b = 4$.

to 1.25, 0.97 and 0.76 (for $\rho = 5, 10, 20$), which, when multiplied by $(\rho - \rho_c)^{1/3}$, consistently with (33), collapse to a value $\simeq 2$.

4.2. Criticality ($\rho = \rho_c$)

The critical case is harder to analyse numerically, because of the presence of large fluctuations in the system, and because the asymptotic regime is longer to attain. Guided by the mean-field analysis, we expect the dynamics to exhibit two scales of time, corresponding to regimes I and II respectively. In the latter, let us assume the scaling behaviour, inspired by (24),

$$f_k(t) \approx f_{k,\mathrm{eq}}\, g_c(u) \qquad u = kt^{-1/z_c}$$

where z_c is the critical dynamical exponent. In order to determine the value of this exponent, we measured

$$\nu_{b,n} = \sum_{k=0}^{\infty} \frac{f_k(t)}{f_{k,\mathrm{eq}}} k^n \sim t^{(1/z_c)(n-1)} \int_0^{\infty} \mathrm{d}u\, g_c(u) \tag{34}$$

for $n = 0$ and $n = 1$, and where the notation recalls the fact that, at large values of k, $f_{k,\mathrm{eq}} \sim k^{-b}$. Figure 5 depicts a plot of $\nu_{b,0}$ for $b = 3, 4, 5$ against $t^{1/5}$, which points towards $z_c = 5$. Actually the best regression fit is obtained for $z_c^{-1} \simeq 0.215$. We checked that $\nu_{b,1} \sim (\nu_{b,0})^2$. The ratio of these two quantities plotted against time gives an indication of how the asymptotic regime is attained. Figure 6 depicts a plot of the critical scaling function $g_c(u)$, against the scaling variable $u = kt^{-1/5}$, for three different times (800, 2000 and 4000).

In regime I the sum rules (1) and (2) yield, along the same lines as in section 3.2 (see equations (25) and (26)),

$$\omega = \frac{b-2}{z_c} \tag{35}$$

which generalizes (27).

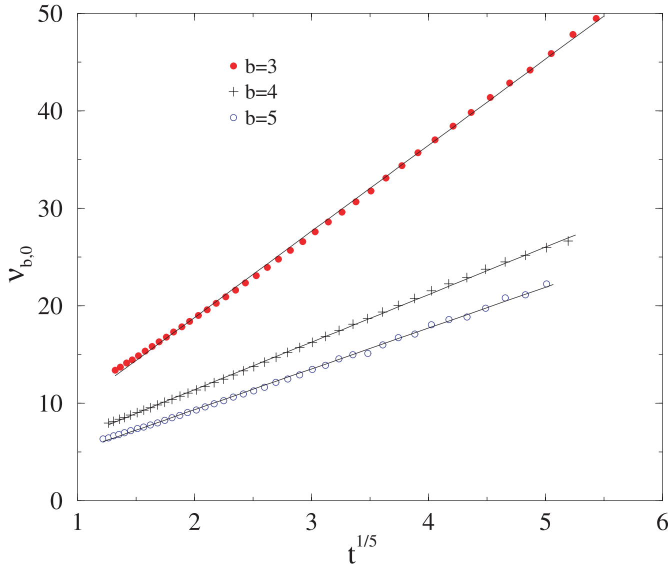

Figure 5. One-dimensional system: plot of $v_{b,0}$ against $t^{1/5}$ at criticality, for $b = 3, 4, 5$, with regression lines; $\rho = \rho_c = 1/(b-2)$.

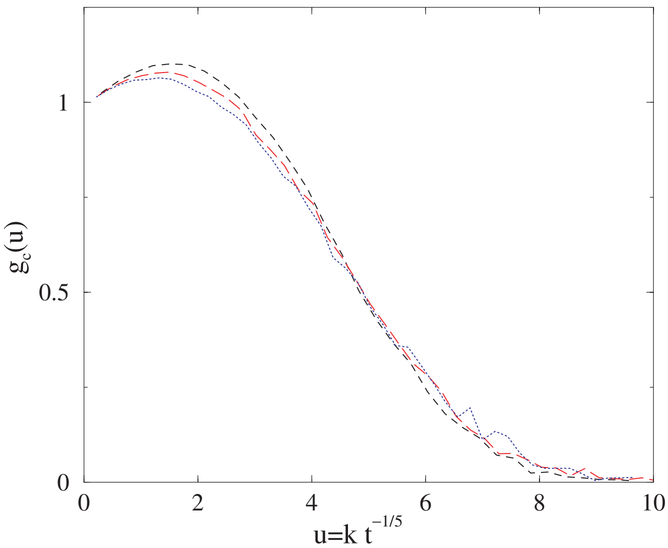

Figure 6. One-dimensional system: plot of the critical scaling function $g_c(u)$ for three different times (800, 2000 and 4000) (dashed, long-dashed, dots); $b = 4$, $\rho = \rho_c = 0.5$.

4.3. Universality

As for the mean-field case, we compared the numerical results obtained with the choice of rates (14), (29) and (30). For instance, in the condensed phase, consider the mean squared occupation μ_2, defined by equation (31). The global appearance of the set of three curves thus obtained is very similar to the corresponding set of curves obtained in the mean-field case, for the same choice of rates. For the latter, one observes weak discrepancies between the curves at shorter times, which however disappear at very large times, say 10^5, obtained by numerical integration of the master equation (17).

The same analysis can be done at criticality, for instance by considering the quantities $\nu_{b,0}$ and $\nu_{b,1}$ defined in (34), with the same conclusions. In particular, the ratio $\nu_{b,1}/(\nu_{b,0})^2$ converges at large times to a universal amplitude ratio only dependent on b.

We are thus led to infer that, for the one-dimensional system, the scaling functions describing the asymptotic behaviour of the occupation probabilities f_k are, as for the mean-field system, universal quantities, independent of the detailed form of the hopping rate d_k, provided that at large values of k, $d_k \approx 1 + b/k$. However, in contrast with the mean-field case, in the condensed phase there exists a dependence on the density through a rescaling of time.

5. Discussion

5.1. Universality: a summary and the role of a bias

Much emphasis was given in the present work to universal aspects of dynamics. Numerical simulations demonstrate the existence of scaling in the one-dimensional system, very akin to the scaling behaviour present at the mean-field level. The scaling regime is characterized by scaling functions, g and g_c, in the condensed phase and at criticality respectively, which do not depend on the detailed form of the rate d_k, provided that at large values of the occupation variable, $d_k \approx 1 + b/k$. They only depend on the parameter b, and, in the condensed phase, on the density, in contrast with the mean-field case. However, a rescaling of time by the density, and correspondingly of g, leads to excellent data collapse. This universality extends to the case of the mean-field zeta urn model, though the definition of the model is different from that of the zero-range processes studied here. The dynamical exponents z and z_c are universal in a stronger sense since they neither depend on b, nor on the density. They only depend on the dimensionality of space, with the mean-field geometry corresponding to the limiting case $d = \infty$.

The observed differences between the values of the dynamical exponents z and z_c, in the mean-field geometry on the one hand, and in the one-dimensional case on the other, give a measure of the role of the diffusion of particles from sites to sites. Indeed, besides the fact, independent of the geometry of the system, that the occupation numbers N_i are fluctuating variables by the very definition of the process, for the one-dimensional system a new phenomenon appears, which is the spatial diffusion of particles along the line. This induces correlations between neighbouring boxes (sites).

To gain more understanding of this point, we introduced a bias in the move of a particle out of a box. A particle now hops to the right with probability p and to the left with probability $q = 1 - p$. This does not change the stationary state. For any positive value of the bias $p - q$ we found $z = 2$ and $z_c \simeq 3$ (see figure 7). Therefore the symmetric and asymmetric one-dimensional ZRP belong to two different universality classes. In the extreme case where $p = 1$, corresponding to the totally asymmetric ZRP, there is no diffusion left in the motion of a particle along the line. In other words, once a particle has left a site, it does not revisit this site, as for the mean-field ZRP (for an infinite system). In comparison, for the symmetric case once a particle has left a site, it comes back to this site an infinite number of times: the random walk experienced by a particle is recurrent. How far can we pursue this analogy between the

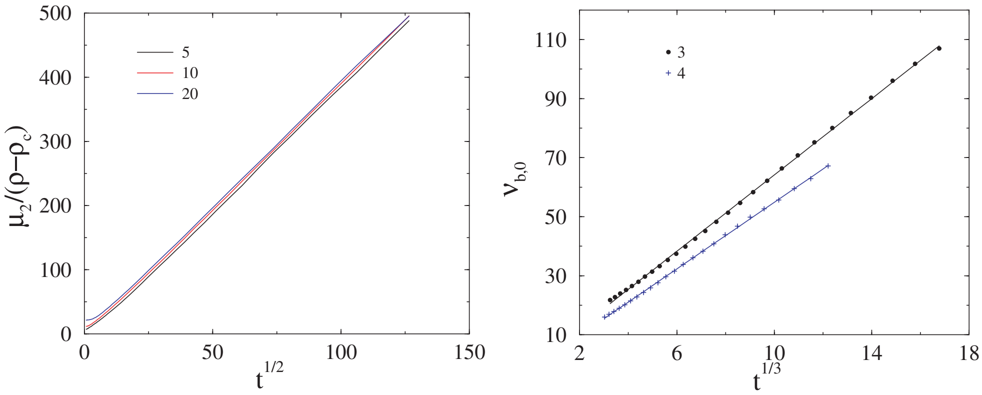

Figure 7. One-dimensional system: asymmetric ZRP ($p = 1$). Left: rescaled mean squared occupation $\mu_2/(\rho - \rho_c)$ against $t^{1/2}$ in the condensed phase, for $\rho = 5$, 10, 20, from bottom to top; $b = 4$. Right: $\nu_{b,0}$ against $t^{1/3}$ at criticality, for $b = 3$, 4, with regression lines; $\rho = \rho_c = 1/(b-2)$.

Table 1. Values of the dynamical exponents for three classes of universality: mean-field geometry ($d = \infty$), one-dimensional asymmetric ($d = 1$, $p \neq \frac{1}{2}$), one-dimensional symmetric ($d = 1$, $p = \frac{1}{2}$).

	$d = \infty$	$d = 1, p \neq \frac{1}{2}$	$d = 1, p = \frac{1}{2}$
z	2	2	3
z_c	2	$\simeq 3$	$\simeq 5$

totally asymmetric ZRP and the mean-field ZRP? It turns out that, in the condensed phase, the scaling function $g(u)$ of the former is quantitatively very close to the scaling function of the latter. Moreover, for $p = 1$, the universality of the scaling function $g(u)$ with respect to density is restored, without rescaling of time. (This point was also checked for the value $p = 3/4$.) However, in spite of the similarities between the asymmetric ZRP and the mean-field ZRP (including the fact that $z = 2$ for both), the fact that, seemingly, $z_c \simeq 3$ for the former case, while $z_c = 2$ for the latter, is a caveat that the influence of a bias is more subtle. As a first step, it would be interesting to have a better understanding of the role of the bias on the decay of correlations.

A summary of the different cases studied in this work is presented in table 1. As indicated by the $\simeq$ sign, in one dimension the status of the growth exponent z and of the dynamical critical exponent z_c are different. The predictions concerning the growth exponents ($z = 3$ and $z = 2$) are well established. They are based on numerical simulations of high quality, for several values of the parameter b, of the density, and for long times. Furthermore, there is an overall coherence of the results (see the discussion above on the recurrence of random walkers, and the discussion below on the analogy with Kawasaki dynamics). The predictions concerning the dynamical critical exponents are less secure. At criticality, large fluctuations, and critical slowing down, lead to a more difficult numerical study. The values given in the table are therefore just indicative. One can wonder whether the critical exponents should have simple values (integer or rational) or not. For simplicity, we reported the closest integer values for the measured exponents in the table. In any case, one should bear in mind that the model studied is one dimensional, with independent sites, and therefore complicated values for the exponents would be rather surprising.

5.2. Future work

Finding an analytical argument in favour of the values of the exponents appearing in the table is the most immediate challenge. Other immediate points of interest are the determination of the upper critical dimension for the zero-range processes under study and the explanation of the role of the density in the one-dimensional condensed phase.

The value $z = 3$ is reminiscent of the value of the dynamical exponent for the growth of domains in the dynamics of an Ising chain under Kawasaki dynamics, in the scaling regime of low temperature [27, 28]. It is interesting to note that, for the Kawasaki chain, adding a bias induces a change in the dynamic exponent from $z = 3$ to $z = 2$ [29, 30]. This difference of one unit in the value of the exponent z, when going from the symmetric case to the asymmetric one, lies in the fact that, for the former case, the probability that a spin detaching from a domain reaches the neighbouring one is inversely proportional to the distance between domains, according to the classical gambler's ruin problem [31], while this no longer holds in presence of a bias, where moves of a spin are ballistic. In this respect, there is some similarity between the dynamics of the Ising chain under Kawasaki dynamics at low temperature and that of the one-dimensional ZRP studied here (see the discussion in section 5.1).

Another point of important interest concerns non-stationary aspects of the dynamics. Non-stationarity is demonstrated by the behaviour of two-time quantities such as two-time autocorrelation, response and fluctuation–dissipation ratio. At the mean-field level, guided once more by the analysis of the dynamics of the zeta urn model [24], we can show that the formalism developed for this model holds for the ZRP studied in the present work. This, as well as the study of the behaviour of two-time quantities for the one-dimensional ZRP, will be the subject of another work [32].

The last challenging point concerns the possible relevance of the present work for the description of the class of driven particle systems exhibiting a condensation transition, such as those studied in [21]. The question posed is whether these particle systems can be described *during their temporal evolution*, and not only at stationarity, in terms of the zero-range processes studied in the present work [22]. As a first step one can use the present study as a template, and ask the following questions [21]. In the condensed phase, or at criticality, is there a scaling regime, and if so, what are the values of the dynamical exponents? What is the role of the density? What is the influence of a bias?

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Note added in proof. After this work was submitted for publication I became aware of the related work of Grosskinsky *et al* [33][2]. In particular, this work confirms the values of the dynamical exponents in the condensed phase, $z = 3$, for the symmetric case, and $z = 2$, for the asymmetric one, which are determined numerically by measuring the size of the largest boxes. The values of the exponents are also estimated by heuristic arguments which provide an explanation to the observed dependence of the mean squared occupation in the density in the condensed phase (see figure 4 of the present work). Finally the ruin problem probability must be taken into account in the argument, for the symmetric case, while it is not in the presence of a bias [33]. This induces, as for the Kawasaki chain at low temperature, a difference of one unit in the value of z.

[2] I am indebted to G Schüz for thorough discussions on [33].

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